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ON CERTAIN REPRESENTATIONS OF POSITIVE INTEGERS

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In this paper we investigate some properties of positive integers n, which are representable in the form $n = ux + vy$, where u and v are two positive and relatively prime integers, and x and y are non negative integers; these integers are called representable or representable by u and v.

The following properties are well known [1]). (Confer the appendix).

All integers $\geq (u - 1)(v - 1)$ are representable by u and v. The integer $N = uv - u - v$ cannot be represented by u and v. If an integer n with $0 \leq n \leq N$ is representable, then $N - n$ is not, and conversely. Hence there are $\frac{1}{2}(u - 1)(v - 1)$ non negative integers[2]) which cannot be represented by u and v.

In what follows P denotes the set of integers which are representable by u and v and which are $\leq N$; Q denotes the set of non negative integers which are not representable by u and v. Then $P \cup Q$ is the set $0, 1, \ldots, N$. Further U denotes the set $1, \ldots, u - 1$ and V denotes the set $1, \ldots, v - 1$.

In order to deduce properties of the elements of P and Q we define for any c and any set M the set $M + c$ as the set of all elements $m + c$ where $m \in M$; further we define the set cM as the set of all elements cm where $m \in M$. Finally we shall denote the sum of the k^{th} powers of the elements of a set M by M^k.

In this paper we derive a formula for Q^k; confer formula (11).

We now prove two lemma's.

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1) Confer A. Brauer, On a problem of partitions, Amer. J. Math. 64 (1942), 299—312.

2) This property was already mentioned by J. J. Sylvester, Math. Questions with their solutions from the educational times, 41 (1884), 21.

L e m m a 1. If $q \in Q$ and $q \notin Q + u$, we have $q \in U$ and $v \nmid q$, and conversely.

P r o o f. Since $q \notin Q + u$, either $q - u$ is representable or $q - u < 0$. If $q - u$ is representable, so is q, which contradicts $q \in Q$. Hence $q < u$. From $q \in Q$ it follows that $q > 0$, so $0 < q < u$, i.e. $q \in U$. Further since $q \in Q$ we have $v \nmid q$.

Conversely if $q \in U$ and $v \nmid q$ the integer q is not representable, so $q \in Q$. Further $q - u < 0$, so $q - u \notin Q$, hence $q \notin Q + u$.

L e m m a 2. If $q \in Q + u$ and $q \notin Q$, we have $q \in vW$, where W denotes the set $\left[\frac{u}{v}\right] + 1, \ldots, u - 1$, and conversely.

P r o o f. Since $q \in Q + u$ we have $q > 0$ and since $q \notin Q$, two non negative integers x and y exist with $q = ux + vy$. Further from $q \in Q + u$ it follows that $q - u \in Q$, so $q - u = u(x - 1) + vy$ is not representable. Now $y \geq 0$, so $x - 1 < 0$, hence $x = 0$ and $q = vy$. Finally from $q \in Q + u$ it follows that $0 < q - u \leq uv - - u - v$, so $u < vy \leq (u - 1)v$. Thus $\left[\frac{u}{v}\right] + 1 \leq y \leq u - 1$ and $q \in vW$.

Conversely since $q \in vW$ obviously $q \notin Q$ and further $q = vy$ with $\left[\frac{u}{v}\right] + 1 \leq y \leq u - 1$. The positive integer $q - u$ is not representable, for otherwise non negative integers x' and y' would exist with $q - u = vy - u = ux' + vy'$, hence $u(x' + 1) = v(y - y')$. This would imply $u | y - y'$ which is impossible, since $0 < y - y' \leq$ $\leq y \leq u - 1$. Hence $q \in Q + u$.

Applying lemma 1 and 2 we get

$$Q \cup (vW) = (Q + u) \cup Z, \tag{1}$$

where Z denotes the set of all elements of U, which are not divisible by v.

If $u < v$, we have $W = Z = U$. If however $u > v$, we add on both sides of (1) the set with elements $v, 2v, \ldots, \left[\frac{u}{v}\right] v$. So in both cases we get

$$Q \cup (vU) = (Q + u) \cup U. \tag{2}$$

By symmetry we also have

$$Q \cup (uV) = (Q + v) \cup V. \tag{3}$$

We now deduce a formula for Q^k for non negative integers k. First we mention a few properties of the polynomials $B_h(x)$ of Bernoulli which enable us to calculate the U^k.

From

$$u^k + U^k = (U + 1)^k + 1 = \Sigma_{h=0}^{k} \binom{k}{h} U^h + 1$$

it follows that

$$\Sigma_{h=0}^{k-1} \binom{k}{h} U^h = u^k - 1. \tag{4}$$

On the other hand we have

$$B_{k+1}(x) - B_{k+1}(x - 1) = (k + 1)\,(x - 1)^k,$$

so

$$U^k = \frac{1}{k + 1} (B_{k+1}(u) - B_{k+1}(1)),$$

hence, using the formula

$$B_{k+1}(x) = \Sigma_{h=0}^{k+1} \binom{k+1}{h} x^h B_{k+1-h} \tag{5}$$

we get

$$U^k = \frac{1}{k + 1} \Sigma_{h=1}^{k+1} \binom{k+1}{h} (u^h - 1) B_{k+1-h} =$$

$$= \frac{1}{k + 1} \Sigma_{t=0}^{k} \binom{k+1}{t+1} (u^{t+1} - 1) B_{k-t}. \tag{6}$$

We can interpret our result as follows. From the equation (4) taken for $k = 1, \ldots, K$, which equation is linear in the unknowns $U^0, \ldots, U^{K-1}$ these unknowns can be found and obviously are a linear compositum of the right hand members $u - 1$, $u^2 - 1, \ldots$, $u^K - 1$ of the equations (4). These values of the unknowns are given by (6).

These results are used now to determine Q^k. Taking the sum of the k^{th} powers of all elements in both sides of the formula (2) we get, since $Q \cap (vU) = (Q + u) \cap U$ is empty, the relation

$$Q^k + v^k U^k = (Q + u)^k + U^k$$

hence

$$\Sigma_{h=0}^{k-1} \binom{k}{h} u^{k-h} Q^h = (v^k - 1) U^k,$$

so

$$\Sigma_{h=0}^{k-1} \binom{k}{h} \frac{Q^h}{u^h} = \frac{v^k - 1}{u^k} U^k. \tag{7}$$

Now if in the equations (4) we replace the unknowns U^h by $\dfrac{Q^h}{u^h}$ and the right hand sides $u^k - 1$ by $\dfrac{v^k - 1}{u^k}\, U^k$, we obtain the equations (7). Hence by the above remark the values of $\dfrac{Q^h}{u^h}$ must be found from (6) by the same substitution i.e.

$$\frac{Q^k}{u^k} = \frac{1}{k+1} \Sigma_{t=0}^{k} \binom{k+1}{t+1} \frac{v^{t+1} - 1}{u^{t+1}} U^{t+1} B_{k-t},$$

and substituting in this last result for U^{t+1} its value given by (6) we get

$$Q^k = \frac{1}{k+1} \Sigma_{t=0}^{k} \binom{k+1}{t+1} (v^{t+1} - 1) u^{k-t-1} B_{k-t} \frac{1}{t+2} \Sigma_{s=0}^{t+1} \binom{t+2}{s+1} .$$

$$(u^{s+1} - 1)\, B_{t+1-s}. \tag{8}$$

To reduce the last member of (8) we first calculate the expression

$$\frac{1}{k+1} \Sigma_{t=0}^{k} \binom{k+1}{t+1} (v^{t+1} - 1) u^{k-t-1} B_{k-t} \frac{1}{t+2} \Sigma_{s=0}^{t+1} \binom{t+2}{s+1} B_{t+1-s}. \tag{9}$$

Now we have from (5) with $x = 1$

$$\Sigma_{h=0}^{t+2} \binom{t+2}{h} B_{t+2-h} = B_{t+2}(1),$$

so

$$\Sigma_{h=1}^{t+2} \binom{t+2}{h} B_{t+2-h} = B_{t+2}(1) - B_{t+2}(0) = 0$$

since $t + 1 \geq 1$. Thus the expression $\Sigma_{s=0}^{t+1} \binom{t+2}{s+1} B_{t+1-s}$ vanishes and so does (9). Hence (8) reduces to

$$Q^k = \frac{1}{k+1} \Sigma_{t=0}^{k} \binom{k+1}{t+1} (v^{t+1} - 1) u^{k-t-1} B_{k-t} \frac{1}{t+2} .$$

$$. \Sigma_{s=0}^{t+1} \binom{t+2}{s+1} u^{s+1} B_{t+1-s} =$$

$$= \frac{1}{k+1} \Sigma_{t=-1}^{k} \Sigma_{s=0}^{t+1} \binom{k+1}{t+1} \binom{t+2}{s+1} \frac{1}{t+2} (v^{t+1}—1) u^{k+s-t} B_{k-t} B_{t+1-s},$$

where in the first sum the term with $t = -1$ which vanishes, has been added. Putting $k - t = i$, $t + 1 - s = j$ we get

$$Q^k = \frac{1}{k+1} \Sigma_{i=0}^{k+1} \Sigma_{j=0}^{k+1-i} \binom{k+1}{i} \binom{k-i+2}{j} \frac{B_i B_j}{k-i+2} (v^{k-i+1}—1) u^{k-j+1} =$$

$$\Sigma_{\substack{i,j \geqq 0 \\ i+j \leqq k+1}} \frac{k! B_i B_j}{i! j! (k+2-i-j)!} v^{k-i+1} u^{k-j+1} - C, \qquad (10)$$

where

$$C = \Sigma_{\substack{i,j \geqq 0 \\ i+j \leqq k+1}} \frac{k! B_i B_j}{i! j! (k+2-i-j)!} u^{k-j+1} =$$

$$= k! \, \Sigma_{j=0}^{k+1} \frac{B_j u^{k-j+1}}{j!} \Sigma_{i=0}^{k+1-j} \frac{B_i}{i! (k+2-i-j)!} =$$

$$= k! \, \Sigma_{j=0}^{k+1} \frac{B_j u^{k-j+1}}{j!} \frac{B_{k+2-j}(1) - B_{k+2-j}}{(k+2-j)!}.$$

Here we used (5) with $x = 1$ and $k + 2 - j$ instead of $k + 1$.

Now for $k + 2 - j > 1$ we have $B_{k+2-j}(1) = B_{k+2-j}$ and for $k + 2 - j = 1$ we have $B_{k+2-j}(1) = B_{k+2-j} + 1$. So we find

$$C = k! \frac{B_{k+1}}{(k+1)!} = \frac{B_{k+1}}{k+1}$$

and then from (10) we get

T h e o r e m. If Q^k denotes the sum of the k^{th} powers of the non negative integers which are not representable by u and v, then we have

$$Q^k = \Sigma_{\substack{i,j \geqq 0 \\ i+j \leqq k+1}} \frac{k! B_i B_j}{i! j! (k+2-i-j)!} v^{k-i+1} u^{k-j+1} - \frac{B_{k+1}}{k+1}. \qquad (11)$$

This result may symbolically be written in the form

$$Q^k = \frac{u^{k+1} v^{k+1}}{(k+1)(k+2)} \left\{ \left(1 + \frac{B}{u} + \frac{B}{v}\right)^{k+2} - \left(\frac{B}{u} + \frac{B}{v}\right)^{k+2} \right\} - \frac{B_{k+1}}{k+1},$$

where in the ordinary expansion of the $(k+2)^{th}$ powers instead of B^h has to be taken B_h.

If we take $k = 0$ we find the above formula $Q^0 = \frac{1}{2}(u—1)(v—1)$ for the number of elements of Q.

A p p e n d i x. Above we used some results of which easily a proof is given by the following considerations.

Let as before u and v denote two integers > 1 with $(u, v) = 1$. Let $\binom{n}{u,\, v}$ denote the number of different ways in which the integer n can be written in the form $n = ux + vy$ with non negative integers x and y. Then obviously

$$\frac{1}{(1 - z^u)\,(1 - z^v)} = \Sigma_{n=0}^{\infty} \binom{n}{u,\, v} z^n.$$

Since $(u, v) = 1$ the expression

$$\frac{(1 - z^{uv})\,(1 - z)}{(1 - z^u)\,(1 - z^v)}$$

is a polynomial in z of degree $N + 1$ where $N = uv - u - v$. Hence we have

$$\frac{(1 - z^{uv})\,(1 - z)}{(1 - z^u)\,(1 - z^v)} = \Sigma_{n=0}^{N+1} \binom{n}{u,\, v} z^n - \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^{n+1} =$$

$$= (1 - z)\, \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^n + \binom{N+1}{u,\, v} z^{N+1}.$$

Obviously the coefficient of z^{N+1} in the expansion is equal to 1, so

$$\frac{1 - z^{uv}}{(1 - z^u)\,(1 - z^v)} = \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^n + \frac{z^{N+1}}{1 - z}. \tag{12}$$

Replacing z by $\frac{1}{z}$ and multiplying by z^N we get

$$\frac{z^{uv} - 1}{(z^u - 1)\,(z^v - 1)} = \Sigma_{n=0}^{N} \binom{n}{u,\, v} z^{N-n} + \frac{1}{z - 1} = \Sigma_{n=0}^{N} \binom{N-n}{u,\, v} z^n + \frac{1}{z - 1}. \tag{13}$$

Comparing (12) and (13) we get for $n = 0, 1, \ldots, N$

$$\binom{n}{u,\, v} + \binom{N - n}{u,\, v} = 1.$$

Since for all n we have $\binom{n}{u,\, v} \geq 0$, we get for $n = 0, 1, \ldots, N$ the result $\binom{n}{u,\, v} = 0$ or 1, so all these integers n are either not representable or are representable in exactly one way. Further we get from (12)

$$\Sigma_{n=0}^{\infty}\binom{n}{u,\, v} z^n = \frac{1}{(1 - z^u)\,(1 - z^v)} =$$

$$= \frac{z^{N+1}}{1 - z} + \frac{z^{uv}}{(1 - z^u)\,(1 - z^v)} + \Sigma_{n=0}^{N}\binom{n}{u,\, v} z^n$$

where for $n \geq N + 1$ the coefficient of z^n in the right hand side is obviously ≥ 1. So every integer $n \geq N + 1$ is representable.

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